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Anti-*G*-Hermiticity Preserving Linear Map That Preserves Strongly the Invertibility of Calkin Algebra Elements

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ABSTRACT

A linear map $\psi : X \to Y$ of algebras X and Y preserves strongly invertibility if $\psi(x^{-1}) = \psi(x)^{-1}$ for all $x \in X^{-1}$, where X^{-1} denotes the set of invertible elements of X. Let $B(H)$ be the Banach algebra of all bounded linear operators on a separable complex Hilbert space H with dim $H = \infty$. A Calkin algebra $C(H)$ is the quotient of $B(H)$ by $K(H)$, the ideal of compact operators on H. An element $A + K(H) \in C(H)$ is said to be anti-G-Hermitian if $(A + K(H))^{\#} = -A + K(H)$, where the $^{\#}$-operation is an involution on $C(H)$. A linear map $\tau : C(H) \to C(H)$ preserves anti-G-Hermiticity if $\tau\,(A + K(H))^{\#} = -\tau\,(A + K(H))$ for all anti-G-Hermitian element $A + K(H) \in C(H)$. In this paper, we characterize the continuous unital linear map $\tau : C(H) \to C(H)$ induced by the essentially anti-G-Hermiticity preserving linear map $\phi : B(H) \to B(H)$ that preserves strongly the invertibility of operators on H. We also take a look at the linear preserving properties of this induced map and other linear preservers on $C(H)$. The discussion is in the context of G-operators, that is, linear operators on H with respect to a fixed but arbitrary positive definite Hermitian operator $G \in B(H)^{-1}$.

Keywords: Linear preservers, Calkin algebra, inner product, anti-Hermiticity, invertibility

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INTRODUCTION

Let $B(S)$ denote the algebra of all bounded linear operators on a Banach space S. A *linear preserver* is a linear map of an algebra X into another algebra Y which preserves certain properties of some elements of X into Y, and a *linear preserver problem* is an area in operator theory which is concerned with the characterization of such maps. An example of a linear preserver is the continuous bijective linear map ϕ: $B(S) \to B(S)$, which preserves commutativity on a complex Banach space S for which there exists a scalar $c_1 \neq 0$ such that for every idempotent operator $P \in B(S)$ of rank 1, there is an idempotent operator $P_0 \in B(S)$ of rank 1 and a scalar c_2 such that $\phi\ (P) = c_1 P_0 + c_2 I$, where I is the identity operator (Omladič, 1986). In the course of the study of linear preservers, there are some characterizing properties that may be obtained which involve automorphisms or anti-automorphisms. These types of linear maps often occur in the results of linear preserver problems because they certainly preserve several properties of the elements of a Banach space or a Banach algebra (Chahbi et al., 2015). Examples of these are the continuous bijective linear map $\phi : B(S) \to B(S)$ that preserves spectrum on a finite dimensional Banach space S, which turns out to be either an automorphism or an anti-automorphism (Jafarian & Sourour, 1986), and the bijective continuous linear map $\phi : B(H) \to B(H)$ that preserves projection operators on Hilbert space H, which turns out to be either a *-automorphism or a *-anti-automorphism (Brešar & Šemrl, 1997).

Let X, Y be algebras, and let $\psi : X \to Y$ be a linear map. We say that ψ *preserves invertibility* if $\psi(x) \in Y^{-1}$ for all $x \in X^{-1}$, where X^{-1} and Y^{-1} denote the set of invertible elements of X and Y, respectively. Further, a linear map $\psi : X \to Y$ *preserves strongly invertibility* if $\psi(x^{-1}) = \psi(x)^{-1}$ for all $x \in X^{-1}$. This paper takes a look at the continuous unique linear map $\tau : C(H) \to C(H)$ induced by the linear map $\phi : B(H) \to B(H)$, which preserves essentially anti-G-Hermiticity and preserves strongly the invertibility of operators on H, where H is a separable complex Hilbert space with $\dim H = \infty$, and $C(H)$ is the Calkin algebra of H. We focus on the characterization of the anti-G-Hermiticity preserving linear map τ that preserves strongly the invertibility of the elements of $C(H)$. This paper also seeks to investigate the other related linear preservers on $C(H)$. The discussion is in the context of G-operators on H, that is, linear operators on H with respect to a fixed but arbitrary positive definite Hermitian operator $G \in B(H)^{-1}$.

PRELIMINARIES

The notion on G-operators was conceived from linear operators on a Krein space. The first works on linear operators on an infinite-dimensional Krein space started with the groundbreaking paper by Pontryagin. Since then, the theory of linear operators on Krein spaces has been developed into a major branch of modern operator theory. In a Krein space K, there is a defined sesquilinear form $[\cdot,\cdot]$ induced by a Hermitian operator $J : K \to K$ such that $J^2 = I$ and $[x,y] = \langle Jx, y\rangle$ for all $x,y \in K$ (Azizov & Iokhvidov, 1989).

Consider the complex Hilbert space H with its corresponding inner product $\langle\cdot,\cdot\rangle$, that is, $(H, \langle\cdot,\cdot\rangle)$. With respect to G, define the mapping $[\cdot,\cdot]_G : H \times H \to \mathbb{C}$ such that $[\cdot,\cdot]_G\ (x, y) = [x, y]_G = \langle Jx, y\rangle$, for all $x, y \in H$ (Bognar, 1974). Since $[\cdot,\cdot]_G$ satisfies all the conditions of an inner product, and H is a complete normed space and $[\cdot,\cdot]_G$ is expressible in terms of $\langle\cdot,\cdot\rangle$, and H is also complete with respect to the new inner product $[\cdot,\cdot]_G$, we say that H is also a Hilbert space with respect to $[\cdot,\cdot]_G$. Thus, for all $x, y \in H$ and for all $A \in B(H)$,

$$\begin{aligned}[Ax, y]_G &= \langle GAx, y\rangle \\ &= \langle x, A^*Gy\rangle \\ &= \langle (G^{-1})^*Gx, A^*Gy\rangle \\ &= \langle Gx, G^{-1}A^*Gy\rangle \\ &= [x, G^{-1}A^*Gy]_G.\end{aligned}$$

The adjoint of A with respect to the new inner product $[\cdot,\cdot]_G$ is called the *G-adjoint* of A, denoted by $A^\#$, where $[Ax, y]_G = [x, A^\# y]_G$. Hence, $A^\# = G^{-1}A^*G$ for all $A \in B(H)$. In this paper, we present a discussion of operators on $(H, [\cdot,\cdot])$, which is analogous to the discussion of operators on $(H, \langle\cdot,\cdot\rangle)$.

Let $K(H)$ be the ideal of all compact operators on H. The quotient of $B(H)$ by $K(H)$ given by the collection

$$C(H) = \{A + K(H) : A \in B(H)\}$$

is called the *Calkin algebra* of $B(H)$ by $K(H)$. In the Calkin algebra $C(H)$, addition, operator multiplication, and scalar multiplication operations follow from that of the cosets of ideals and that for all $n \in \mathbb{N}$, $(A + K(H))^n = A^n + K(H)$. The zero element in $C(H)$ is $K(H)$, and the unit element is $I + K(H)$. If $A \in B(H)^{-1}$, then the inverse of $A + K(H)$ is $A^{-1} + K(H)$. In general, if A is not necessarily invertible, the inverse of $A + K(H)$ is the Calkin algebra element $B + K(H)$ such that $AB - I, BA - I \in C(H)$. It can be noted that based on the concepts on cosets, we can say that $A + K(H) = K(H)$ if and only if $A \in K(H)$, and also, $A_1 + K(H) = A_2 + K(H)$ if and only if $A_1 - A_2 \in K(H)$. Both $B(H)$ and $K(H)$ are C^*-algebras and as a quotient of two C^*-algebras, $C(H)$ is a C^*-algebra itself. In $C(H)$, there is an involution $A + K(H) \to (A + K(H))^*$, and the **-operation* on the elements of $C(H)$ is defined by $(A + K(H))^* = A^* + K(H)$, where A^* is the adjoint of A with respect to the inner product $\langle\cdot,\cdot\rangle$ on H.

THE CANONICAL MAP

This section focuses on the continuous canonical mapping from $B(H)$ into $C(H)$. We shall examine the preserving properties of this mapping on some classes of G-operators on H.

Definition 3.1. An operator $V \in B(H)$ is said to be G-Hermitian if $V^\# = V$, and an operator $U \in B(H)$ is said to be anti-G-Hermitian if $U^\# = -U$.

Definition 3.2. An element $V + K(H)$ of the Calkin algebra $C(H)$ is said to be *Hermitian* if $(V + K(H))^* = V + K(H)$, and an element $U + K(H) \in C(H)$ is said to be *anti-Hermitian* if $(U + K(H))^* = -(U + K(H))$.

Definition 3.3. The continuous function $\rho : B(H) \to C(H)$ such that $\rho(A) = A + K(H)$ for all $A \in B(H)$ is called the *canonical map* of $B(H)$ into $C(H)$.

Proposition 3.4. *The canonical map* $\rho : B(H) \to C(H)$ *is a unital linear map.*

Proof. Let $A_1, A_2 \in B(H)$ and $c \in \mathbb{C}$. Then $\rho(A_1 + cA_2) = (A_1 + cA_2) + K(H) = (A_1 + K(H)) + c(A_2 + K(H)) = \rho(A_1) + c\rho(A_2)$. Note that $\rho(I) = I + K(H)$, where $I + K(H)$ is the unit element in $C(H)$. Hence, ρ is a unital linear map.

Remark 3.5. $\rho(G)$ and $\rho(G^{-1})$ are Hermitian elements in $C(H)$.

Definition 3.6. Let $A + K(H) \in C(H)$. We define the *$^\#$-operation* on the elements of $C(H)$ by $(A + K(H))^\# = (G + K(H))^{-1}(A + K(H))^*(G + K(H))$.

Equivalently, in terms of the canonical map ρ, the above definition tells us that for all $A \in B(H)$, $\rho(A)^\# = \rho(G)^{-1}\rho(A)^*\rho(G)$. The next proposition gives us a convenient form of the $^\#$-operation on the elements of $C(H)$.

Proposition 3.7. *For all* $A \in B(H)$,

$$(A + K(H))^\# = A^\# + K(H).$$

The proof of Proposition 3.7 is just a consequence of Definition 3.6. Corollaries

3.8 and 3.9 present some properties of the $^{\#}$-operation on the elements of $C(H)$.

Corollary 3.8. *Let* $A + K(H) \in C(H)$. *Then*

(i) $[(A + K(H))^{\#}]^{-1} = [(A + K(H))^{-1}]^{\#}$ *for all* $A \in \mathrm{B}(H)^{-1}$;

(ii) $[(A + K(H))^{\#}]^{n} = [(A + K(H))^{n}]^{\#}$ *for all* $n \in \mathbb{N}$.

The proof of Corollary 3.8 uses Proposition 3.7 and some properties of the G-adjoint of operators on H. The next result tells us that the map which gives the $^{\#}$-operation on the elements of $C(H)$ is an involution on $C(H)$.

Corollary 3.9. *Let* $A + K(H) \in C(H)$. *Then the map* $A + K(H) \to (A + K(H))^{\#}$ *is an involution on* $C(H)$.

Proof. If $A + K(H) \in C(H)$, then $(A + K(H))^{\#\#} = (A^{\#} + K(H))^{\#} = A + K(H)$ and $[c(A + K(H))]^{\#} = \overline{c}A^{\#} + K(H) = \overline{c}\,(A + K(H))^{\#}$ for any $c \in \mathbb{C}$. If $A_1 + K(H), A_2 + K(H) \in C(H)$, then it can be verified that $[(A_1 + K(H)) + (A_2 + K(H))]^{\#} = (A_1 + K(H))^{\#} + (A_2 + K(H))^{\#}$ and $[(A_1 + K(H))(A_2 + K(H))]^{\#} = (A_2 + K(H))^{\#}(A_1 + K(H))^{\#}$ using Proposition 3.7 and some properties of the G-adjoint of operators on H. Hence, we have shown that the involution $A + K(H) \to (A + K(H))^{*}$ induces an involution $A + K(H) \to (A + K(H))^{\#}$.

We call this $^{\#}$-operation on the elements of the Calkin algebra $C(H)$ in the involution $A + K(H) \to (A + K(H))^{\#}$ the *G-adjoint* of the elements of $C(H)$.

Definition 3.9. An element $V + K(H) \in C(H)$ is said to be *G-Hermitian* if $(V + K(H))^{\#} = V + K(H)$, and an element $U + K(H) \in C(H)$ is said to be *anti-G-Hermitian* if $(U + K(H))^{\#} = -(U + K(H))$.

Equivalently, in terms of the canonical map ρ, the above definition tells us that $\rho(V) \in C(H)$ is G-Hermitian if $\rho(V)^{\#} = \rho(V)$, and $\rho(V) \in C(H)$ is anti-G-Hermitian if $\rho(U)^{\#} = -\rho(U)$. If $\rho(V)$ is G-Hermitian, then $\rho(V)^{n}$ is G-Hermitian for every $n \in \mathbb{N}$ since by Corollary 3.8, $(\rho(V)^{n})^{\#} = (\rho(V)^{\#})^{n} = \rho(V)^{n}$. Also, $\rho(V)^{-1}$ is G-Hermitian since by Corollary 3.8, $(\rho(V)^{-1})^{\#} = (\rho(V)^{\#})^{-1} = \rho(V)^{-1}$. Further, if $\rho(U)$ is anti-G-Hermitian, then $\rho(U)^{n}$ is anti-G-Hermitian for all odd $n \in \mathbb{N}$ since $(\rho(U)^{n})^{\#} = (\rho(U)^{\#})^{n} = (-\rho(U))^{n} = -\rho(U)^{n}$.

Corollary3.10. *If U is anti-G-Hermitian, then $\rho(U)$ is anti-G-Hermitian.*

Corollary 3.11. *If V is G-Hermitian, then $\rho(V)$ is G-Hermitian.*

The converse of Corollary 3.10 is not always true. If $\rho(U)$ is anti-G-Hermitian, then $U^{\#} + U$ is compact, but not necessarily equal to the zero operator O. The converse only holds if $U^{\#} + U = O$. It is also the same case for Corollary3.11, where the converse is not true since if $\rho(V)$ is G-Hermitian, then $V^{\#} - V$ is compact, but not necessarily equal to O. The converse only holds if $V^{\#} - V = O$. From these observations, we can conclude that $U + K(H)$ is an anti-G-Hermitian element of $C(H)$ if and only if $U^{\#} + U$ is compact on H, and also, $V + K(H)$ is a G-Hermitian element of $C(H)$ if and only if $V^{\#} - V$ is compact on H.

The following result summarizes the preserving properties of the canonical map ρ.

Theorem 3.12. *The continuous canonical map* $\rho : B(H) \to C(H)$

(i) preserves strongly invertibility;

(ii) preserves anti-G-Hermiticity;

(iii) preserves G-Hermiticity;

(iv) is a $^{\#}$-epimorphism.

Proof. For (i), let $A \in B(H)^{-1}$. Then $\rho\,(A^{-1}) = A^{-1} + K(H) = (A + K(H))^{-1} = \rho(A)^{-1}$. The proofs of (ii) and (iii) follow from Corollaries 3.10 and 3.11. The surjectivity of ρ in (iv) follows from the definition of ρ. If $A_1, A_2 \in B(H)$, then $\rho(A_1A_2) = A_1A_2 + K(H) = (A_1 + K(H))(A_2 + K(H)) = \rho(A_1)\rho(A_2)$. By Proposition 3.7, $\rho(A^{\#}) = \rho(A)^{\#}$ for all $A \in B(H)$ so that ρ is a $^{\#}$-epimorphism.

LINEAR MAP PRESERVING ESSENTIALLY ANTI-*G*-HERMITIAN OPERATORS ON HILBERT SPACE

In this section, we introduce the notion of essentially G-Hermitian and essentially anti-G-Hermitian operators on Hilbert space H and show that an essentially anti-G-Hermiticity preserving linear map $\phi : B(H) \to B(H)$ preserves compact operators.

Definition 4.1. A linear map $\phi : B(H) \to B(H)$ is said to be *unital up to compact operators* if $\phi(I) = I + K$ for some $K \in K(H)$.

Definition 4.2. A linear map $\phi : B(H) \to B(H)$ is said to be *surjective up to compact operators* if $B(H) = Ran\, \phi + K(H)$, that is, for all $B \in B(H)$, there exists $A \in B(H)$ such that $B = \phi(A) + K$ for some $K \in K(H)$.

Equivalently, the above definition tells us that ϕ is surjective up to compact operators if the composition map $\rho\phi : B(H) \to C(H)$ is onto.

Definition 4.3. A linear map $\phi : B(H) \to B(H)$ is said to be *injective up to compact operators* if for all $A_1, A_2 \in B(H)$, $\phi(A_1) = \phi(A_2)$ implies that $A_1 = A_2 + K$ for some $K \in K(H)$.

Equivalently, the above definition tells us that ϕ is injective up to compact operators if $\phi(A_1) = \phi(A_2)$ implies $\rho(A_1) = \rho(A_2)$.

In our succeeding discussion, whenever we talk about the mapping ρ, we mean the canonical map $\rho : B(H) \to C(H)$ presented in the previous section. It shall be noted that for all $A_1, A_2 \in B(H)$, $\rho(A_1) = \rho(A_2)$ implies that $A_1 = A_2 + K$ for some $K \in K(H)$.

Definition 4.4. An operator $V \in B(H)$ is said to be *essentially G-Hermitian* if $\rho(V)$ is G-Hermitian. An operator $U \in B(H)$ is said to be *essentially anti-G-Hermitian* if $\rho(U)$ is anti-G-Hermitian.

An anti-G-Hermitian operator is essentially anti-G-Hermitian, but an essentially anti-G-Hermitian operator is not necessarily anti-G-Hermitian. The same observation is also true for G-Hermitian and essentially G-Hermitian operators.

Definition 4.5. A linear map $\phi : B(H) \to B(H)$ is said to *preserve essentially G-Hermitian operators* on H if for all essentially G-Hermitian operator V, $\phi(V)$ is essentially G-Hermitian, that is, $\rho(\phi(V))$ is G-Hermitian. A linear map $\phi: B(H) \to B(H)$ is said to *preserve essentially anti-G-Hermitian operators* on H if for all essentially anti-G-Hermitian operator U, $\phi(U)$ is essentially anti-G-Hermitian, that is, $\rho(\phi(U))$ is anti-G-Hermitian.

Proposition 4.6. *Let U be an anti-G-Hermitian operator on H. Then $U + K$, $U - K$, and $U + iK$ are essentially anti-G-Hermitian for any $K \in K(H)$.*

Proof. If $K \in K(H)$, then it can easily be verified that $\rho(U \pm K)^{\#} = -\rho(U \pm K)$ and $\rho(U + iK)^{\#} = -\rho(U + iK)$ for any anti-G-Hermitian operator U on H.

Lemma 4.7. *Let $\phi : B(H) \to B(H)$ be a continuous linear map that preserves essentially anti-G-Hermitian operators on H. Then ϕ preserves compact operators on H.*

Proof. Let U be an anti-G-Hermitian operator and let K be a compact operator on H. By Proposition 4.6, $U \pm K$ is essentially anti-G-Hermitian so that $\phi(U \pm K) = \phi(U) \pm \phi(K)$ is essentially anti-G-Hermitian. Thus, $\rho(\phi(U) \pm \phi(K))^{\#} = -\rho(\phi(U) \pm \phi(K))$ so that $(\phi(U)^{\#} \pm \phi(K)^{\#}) + K(H) = -(\phi(U) \pm \phi(K)) + K(H)$, which implies that $\phi(U)^{\#} + \phi(U) \pm \phi(K)^{\#} \pm \phi(K) \in K(H)$. Since the sum of two compact operators is compact, we get $2\phi(K)^{\#} + 2\phi(K) \in K(H)$ so that $\phi(K)^{\#} + \phi(K) \in K(H)$. Again, by Proposition 4.6, $U + iK$ is essentially anti-G-Hermitian. In a similar manner, it can be shown that $\phi(U)^{\#} + \phi(U) - i\phi(K)^{\#} +$

$i\phi(K) \in K(H)$. It follows that $\phi(K)^\# + \phi(K) + i\phi(K)^\# - i\phi(K) \in K(H)$ so that $\phi(K)^\# - \phi(K) \in K(H)$. Hence, by further manipulation, we get $2\phi(K) \in K(H)$, which implies that $\phi(K) \in K(H)$. Therefore, ϕ preserves compact operators.

The previous lemma says that $\phi(K(H)) \subseteq K(H)$.

Definition 4.8. A linear map $\phi : B(H) \to B(H)$ is said to *preserve the G-adjoint up to compact operators* if for all $A \in B(H)$, $\phi(A^\#) = \phi(A)^\# + K$ for some $K \in K(H)$.

Theorem 4.9. *Let ϕ: $B(H) \to B(H)$ be a continuous linear map. Then the following statements are equivalent:*

(i) ϕ preserves essentially anti-G-Hermitian operators;

(ii) ϕ preserves essentially G-Hermitian operators;

(iii) ϕ preserves compact operators and preserves the G-adjoint up to compact operators.

Proof. (i)⇒(ii): Let V be an essentially G-Hermitian operator on H. Thus, iV is essentially anti-G-Hermitian since $\rho(iV)^\# = -i\rho(V)^\# = -\rho(iV)$. Since ϕ preserves essentially anti-G-Hermitian operators, $\phi(iV)$ is essentially anti-G-Hermitian so that $\rho(\phi(iV))$ is anti-G-Hermitian. Since $\rho(\phi(iV))^\# = -\rho(\phi(iV))$, we get $-i\phi(V)^\# + i\phi(V) \in K(H)$, which implies that $\phi(V)^\# - \phi(V) \in K(H)$ so that $\rho(\phi(V))^\# = \rho(\phi(V))$. Hence, $\phi(V)$ is essentially G-Hermitian. As a result, ϕ preserves essentially G-Hermitian operators.

(ii)⇒(i): Let U be an essentially anti-G-Hermitian. Applying a similar argument in (i)⇒(ii), it can be shown that $\phi(iU)$ is essentially G-Hermitian so that $\rho(\phi(iU))^\# = \rho(\phi(iU))$, which implies that $\rho(\phi(U))^\# = -\rho(\phi(U))$, and thus, $\phi(U)$ is essentially anti-G-Hermitian. Hence, ϕ preserves essentially anti-G-Hermitian operators.

(ii)⇒(iii): Let V be essentially G-Hermitian. Since ϕ preserves essentially G-Hermitian operators on H, $\rho(\phi(V))$ is G-Hermitian so that $\rho(\phi(V))^\# = \rho(\phi(V))$, which implies that $\phi(V)^\# + K(H) = \phi(V) + K(H)$. Let $A \in B(H)$. By Cartesian G-Decomposition, $A = V_1 + iV_2$ for some G-Hermitian operators V_1, $V_2 \in B(H)$. Note that G-Hermitian operators are essentially G-Hermitian. Now, $\rho(\phi(A^\#)) = \phi(A^\#) + K(H) = \phi((V_1 + iV_2)^\#) + K(H) = \phi(V_1 - iV_2) + K(H) = (\phi(V_1) + K(H)) - i(\phi(V_2) + K(H)) = (\phi(V_1)^\# - i\phi(V_2)^\#) + K(H) = \phi(V_1 + iV_2)^\# + K(H) = \rho(\phi(A)^\#)$, that is, $\rho(\phi(A^\#)) = \rho(\phi(A)^\#)$. It follows that $\phi(A^\#) = \phi(A)^\# + K$ for some compact operator K. Hence, ϕ preserves the G-adjoint up to compact operators. Also, by (ii)⇒(i), ϕ preserves essentially anti-G-Hermitian operators, and so, by Lemma 4.7, ϕ preserves compact operators.

(iii)⇒(i): Let U be essentially anti-G-Hermitian. Then $\rho(U)^\# = -\rho(U)$. By Theorem 3.12, ρ is a $^\#$-endomorphism. Thus, ρ preserves the G-adjoint so that $\rho(U^\#) = \rho(-U)$. It follows that there exists $K_1 \in K(H)$ such that $U^\# = -U + K_1$ so that $\phi(U^\#) = \phi(-U + K_1)$. Since ϕ preserves the G-adjoint up to compact operators, there exists $K_2 \in K(H)$ such that $\phi(U)^\# = -\phi(U) + (\phi(K_1) - K_2)$. Since ϕ preserves compact operators, $\phi(K_1)$ is compact. Let $K = \phi(K_1) - K_2$ so that $\phi(U)^\# = -\phi(U) + K$, where $K \in K(H)$. Now, $\rho(\phi(U)^\#) = \rho(-\phi(U) + K)$ so that $\rho(\phi(U))^\# = -\rho(\phi(U))$. Hence, $\rho(\phi(U))$ is anti-G-Hermitian so that $\phi(U)$ is essentially anti-G-Hermitian. Therefore, ϕ preserves essentially anti-G-Hermitian operators on H.

THE INDUCED MAP

In this section, we introduce a map τ: $C(H) \to C(H)$ induced by the essentially anti-G-Hermiticity preserving linear map ϕ : $B(H) \to B(H)$, which is unital and surjective up to compact operators.

Proposition 5.1. *Let ϕ : $B(H) \to B(H)$ be a continuous linear map which is unital and surjective up to compact operators.*

If ϕ preserves essentially anti-G-Hermitian operators, then ϕ induces a continuous unital surjective linear map $\tau : C(H) \to C(H)$ such that $\tau(A + K(H)) = \rho\phi(A)$ for any $A + K(H) \in C(H)$.

Proof. Let $A_1 + K(H)$, $A_2 + K(H) \in C(H)$ such that $A_1 + K(H) = A_2 + K(H)$. Then $A_1 - A_2 \in K(H)$. Applying Lemma 4.7, $\phi(A_1 - A_2) = \phi(A_1) - \phi(A_2) \in K(H)$. Thus, $(\phi(A_1) - \phi(A_2)) + K(H) = K(H)$, which implies that $\rho(\phi(A_1)) = \rho(\phi(A_2))$ so that $\tau(A_1 + K(H)) = \tau(A_2 + K(H))$, and also, if $A_1 + K(H)$, $A_2 + K(H) \in C(H)$ and $c \in \mathbb{C}$, then $\tau[(A_1 + K(H)) + c(A_2 + K(H))] = \tau[(A_1 + cA_1) + K(H)] = \rho\phi(A_1 + cA_1) = \phi(A_1 + cA_1) + K(H) = (\phi(A_1) + K(H)) + c(\phi(A_2) + K(H)) = \rho\phi(A_1) + c\rho\phi(A_2) = \tau(A_1 + K(H)) + c\tau(A_2 + K(H))$, making τ a well-defined linear map.

Since ϕ is unital up to compact operators, there exists $K \in K(H)$ such that $\phi(I) = I + K$ so that $\tau(I + K(H)) = \rho\phi(I) = \phi(I) + K(H) = (I + K(H)) + (K + K(H)) = I + K(H)$. Thus, τ is unital.

Let $B + K(H) \in C(H)$. Then $B \in B(H)$. Since ϕ is surjective up to compact operators, there exists $A \in B(H)$, and there exists $K \in K(H)$ such that $\phi(A) = B - K$. So, there exists $A + K(H) \in C(H)$ such that $B + K(H) = (\phi(A) + K) + K(H) = \phi(A) + K(H) = \rho\phi(A) = \tau(A + K(H))$. As a result, τ is surjective.

For the continuity of τ, since τ is linear, it suffices to show that τ is continuous at $O + K(H) = K(H)$. We shall use the sequential criterion. Now, let $\langle A_n + K(H)\rangle$ be a sequence in $C(H)$ that converges to $K(H)$. Then as $n \to \infty$, $K(H) = \lim(A_n + K(H)) - K(H) = \lim\| A_n + K(H)\|$.

Now, for each n, if $0 = \|A_n + K(H)\| = \text{dist}(A_n, K(H))$, then $A_n \in K(H)$ since $K(H)$ is closed. In this case, set $K_n = O$.

If $0 < \|A_n + K(H)\|$, then since $\|A_n + K(H)\| < 2\|A_n + K(H)\|$, there exists $K_n \in K(H)$ such that $\|A_n + K_n\| \leq 2\|A_n + K(H)\|$, where $\|A_n + K_n\|$ is a norm in $B(H)$ and $\|A_n + K(H)\|$ is a norm in $C(H)$. Thus, $\lim \|A_n + K_n\| = O$ as $n \to \infty$. Hence, $\langle A_n + K_n\rangle$ converges to O in $B(H)$.

Therefore, ϕ induces a continuous surjective unital linear map τ.

In our succeeding discussion, whenever we speak about the map τ, we mean the unique linear map τ induced by ϕ in Proposition 5.1.

Lemma 5.2. *Let $\phi : B(H) \to B(H)$ be an essentially anti-G-Hermiticity preserving linear map which is unital and surjective up to compact operators. Then the linear map τ induced by ϕ*

(i) preserves anti-G-Hermiticity;
(ii) preserves G-Hermiticity; and
(iii) preserves the G-adjoint of the elements of $C(H)$.

Proof. (i) Let $U + K(H) \in C(H)$ be anti-G-Hermitian. Then U is essentially anti-G-Hermitian. Since ϕ preserves essentially anti-G-Hermiticity, $\rho(\phi(U)) = \rho\phi(U)$ is anti-G-Hermitian. It follows that $\tau(U + K(H))^\# = -\tau(U + K(H))$ so that τ preserves anti-G-Hermiticity.

The proof for (ii) follows the same argument to that of (i).

(iii) Let $A + K(H) \in C(H)$. Then $A \in B(H)$. Since ϕ preserves essentially anti-G-Hermitian operators, by Theorem 4.9, $\phi(A^\#) = \phi(A)^\# + K$ for some $K \in K(H)$. Then $\tau((A + K(H))^\#) = \rho\phi(A^\#) = \rho(\phi(A)^\# + K) = \phi(A)^\# + K(H) = (\rho\phi(A))^\# = \tau(A + K(H))^\#$. Hence, τ preserves the G-adjoint of the elements of $C(H)$.

Proposition 5.3. *Let $\phi : B(H) \to B(H)$ be an essentially anti-G-Hermiticity preserving continuous linear map which is unital and surjective up to compact operators.*

(i) If ϕ is a homomorphism, then τ is a $^\#$-epimorphism.
(ii) If ϕ is an anti- homomorphism, then τ is an anti-$^\#$-epimorphism.

Proof. (i) Let $A_1 + K(H)$, $A_2 + K(H) \in C(H)$. Then $A_1, A_2 \in B(H)$. Thus, $\tau((A_1 + K(H))(A_2 + K(H))) = \tau(A_1A_2 + K(H)) = \rho\phi(A_1A_2) = \phi(A_1)\phi(A_2) + K(H) = (\rho\phi(A_1))(\rho\phi(A_2)) = \tau(A_1 +$

$K(H))\ \tau\ (A_2 + K(H))$. The proof for (ii) can be shown in similar manner.

Definition 5.4. A linear map $\phi : B(H) \rightarrow B(H)$ is called a *homomorphism up to compact operators* if for all A, $B \in B(H)$, then $\phi(AB) = \phi(A)\phi(B) + K$ for some $K \in K(H)$. A linear map $\phi : B(H) \rightarrow B(H)$ is called an *anti-homomorphism up to compact operators* if for all A, $B \in B(H)$, then $\phi(AB) = \phi(B)\phi(A) + K$ for some $K \in K(H)$.

Proposition 5.5. *Let $\phi : B(H) \rightarrow B(H)$ be an essentially anti-G-Hermiticity preserving linear map which is unital and surjective up to compact operators.*

(i) If τ is a homomorphism, then ϕ is a homomorphism up to compact operators.

(ii) If τ is an anti-homomorphism, then ϕ is an anti-homomorphism up to compact operators.

Proof. (i) Let A_1, $A_2 \in B(H)$. Then $A_1 + K(H)$, $A_1 + K(H) \in C(H)$. Since τ is a homomorphism, $\tau\ [(A_1 + K(H)\ (A_2 + K(H))] = \tau\ (A_1A_2 + K(H)) = \tau\ (A_1 + K(H))\tau\ (A_2 + K(H))$ so that $\rho\phi\ (A_1A_2) = \rho\phi\ (A_1)\ \rho\phi\ (A_2)$. It follows that $\phi\ (A_1A_2) + K(H) = \phi\ (A_1)\ \phi\ (A_2) + K(H)$, which implies that $\phi\ (A_1A_2) - \phi\ (A_1)\ \phi\ (A_2) \in K(H)$. Thus, $\phi\ (A_1A_2) = \phi\ (A_1)\ \phi\ (A_2) + K$ for some $K \in K(H)$. Hence, ϕ is a homomorphism up to compact operators.

The proof for (ii) can be shown in a similar manner.

ANTI-G-HERMITICITY PRESERVING LINEAR MAP THAT PRESERVES STRONGLY THE INVERTIBILITY OF CALKIN ALGEBRA ELEMENTS

This section presents the characterization of the linear map $\tau : C(H) \rightarrow C(H)$ induced by $\phi : B(H) \rightarrow B(H)$ in the previous section.

Let $\phi : B(H) \rightarrow B(H)$ be an essentially anti-G-Hermiticity preserving linear map that is unital and surjective up to compact operators. If $A + K(H) \in C(H)^{-1}$ such that $A \in B(H)^{-1}$, and if ϕ preserves strongly invertibility, then $\tau\ [(A + K(H))^{-1}] = \rho\phi\ (A^{-1}) = \tau\ (A + K(H))^{-1}$ so that τ preserves strongly invertibility.

Definition 6.1. A linear map $\phi : B(H) \rightarrow B(H)$ is said to *preserve strongly invertibility up to compact operators* if for all $A \in B(H)$, $\phi\ (A^{-1}) = \phi\ (A)^{-1} + K$ for some $K \in K(H)$.

The converse of the discussion before Definition 6.1 is not necessarily true. However, it does hold if ϕ preserves strongly invertibility up to compact operators. Thus, for any $A + K(H) \in C(H)^{-1}$ such that $A \in B(H)^{-1}$, it is easy to see that τ preserves strongly invertibility if and only if ϕ preserves strongly invertibility up to compact operators. The two earlier discussions only hold for an invertible Calkin algebra element $A + K(H)$ for which A is invertible. A more general case where A may not be invertible is presented in Lemma 6.2.

Lemma 6.2. *Let $\phi : B(H) \rightarrow B(H)$ be an essentially anti-G-Hermiticity preserving continuous linear map which is unital and surjective up to compact operators. Then the linear map τ induced by ϕ preserves strongly invertibility if and only if for all A, $B \in B(H)$ such that $AB - BA$, $AB - I \in K(H)$, ϕ is a homomorphism up to compact operators.*

Proof. Let A, $B \in B(H)$ such that $AB - BA$ and $AB - I$ are compact. Then $\rho(AB) = \rho(BA) = \rho(I)$. By Theorem 3.12, $\rho(A)\rho(B) = \rho(B)\rho(A) = \rho(I)$ so that $\rho(A)^{-1} = \rho(B)$. Observe that $(\phi\ (A) + K(H))^{-1} = \phi\ (B) + K(H)$, that is, $I + K(H) = (\phi\ (A) + K(H))\ (\phi\ (B) + K(H)) = \phi\ (A)\ \phi\ (B) + K(H)$. But $I + K(H) = \phi\ (AB) + K(H)$. Hence, $\phi(AB) - \phi(A)\ \phi\ (B) \in K(H)$ so that ϕ is a homomorphism up to compact operators.

Conversely, let $C + K(H) \in C(H)^{-1}$, where $C \in B(H)$. Then there exists $D + K(H) \in C(H)$ such that $(C + K(H))^{-1} = D + K(H)$. Thus,

$I + K(H) = CD + K(H) = DC + K(H)$ so that $CD - DC$, $CD - I \in K(H)$. It follows that $I + K(H) = \tau(C + K(H))\, \tau(D + K(H))$ so that $\tau(C + K(H))^{-1} = \tau(D + K(H)) = \tau[(C + K(H))^{-1}]$. Therefore, τ preserves strongly invertibility.

Corollary 6.3. *Let $\phi : B(H) \rightarrow B(H)$ be an essentially anti-G-Hermiticity preserving continuous linear map which is unital and surjective up to compact operators. If ϕ is a homomorphism or an anti-homomorphism, then the linear map τ induced by ϕ preserves strongly the invertibility of the elements of $C(H)$.*

Proof. Let $A, B \in B(H)$ such that $AB - BA$, $AB - I \in K(H)$.

If ϕ is a homomorphism, then ϕ is clearly a homomorphism up to compact operators by choosing $K = O$.

If ϕ is an anti-homomorphism, then $\phi(AB) - \phi(A)\,\phi(B) = \phi(AB) - \phi(BA) = \phi(AB - BA)$. Since ϕ preserves essentially anti-G-Hermitian operators, by Lemma 4.7, $\phi(AB - BA)$ is compact. So, $\phi(AB) - \phi(A)\,\phi(B) = K$ for some $K \in K(H)$ so that ϕ is a homomorphism up to compact operators.

By Lemma 6.2, ϕ preserves strongly invertibility.

Theorem 6.4. (Mbekhta, 2007) *The continuous linear map $\phi : B(H) \rightarrow B(H)$ is either an automorphism or an anti-automorphism if and only if there exists an invertible operator $T \in B(H)$ such that ϕ takes one of the following forms:*

$$\phi(A) = TAT^{-1} \text{ or } \phi(A) = TA^{t}T^{-1}$$

for all $A \in B(H)$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal basis of H. In particular, ϕ is unital, bijective, and continuous.

Theorem 6.5. (Mbekhta &Boudi, 2010) *Let X and Y be Banach algebras, and let $\phi : X \rightarrow Y$ be an additive map. Then, ϕ preserves strongly invertibility if and only if $\phi(1)\phi$ is a unital Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ, where 1 is the unit element of X.*

Proposition 6.6. *Let $\phi : B(H) \rightarrow B(H)$ be a continuous unital linear map. Then ϕ preserves strongly invertibility if and only if there exists $T \in B(H)^{-1}$ such that for all $A \in B(H)$, either*

$$\phi(A) = TAT^{-1} \text{ or } \phi(A) = TA^{t}T^{-1},$$

where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal basis of H.

Proof.($\Rightarrow$) : Since ϕ is unital and preserves strongly invertibility, by Theorem 6.5, $\phi(I)\phi = I\phi = \phi$ is a Jordan endomorphism. Note that every Jordan homomorphism on a prime algebra is an automorphism or an anti-automorphism. Since $B(H)$ is a prime algebra, ϕ is an automorphism or an anti-automorphism. Then by Theorem 6.4, there exists an invertible operator $T \in B(H)$ such that ϕ takes one of the following forms:

$$\phi(A) = TAT^{-1} \text{ or } \phi(A) = TA^{t}T^{-1}$$

for all $A \in B(H)$.

($\Leftarrow$) : Suppose there exists $T \in B(H)^{-1}$ such that for all $A \in B(H)$, either $\phi(A) = TAT^{-1}$ or $\phi(A) = TA^{t}T^{-1}$ for all $A \in B(H)$. If $\phi(A) = TAT^{-1}$, then $\phi(A^{-1}) = TA^{-1}T^{-1} = (TAT^{-1})^{-1} = \phi(A)^{-1}$. If $\phi(A) = TA^{t}T^{-1}$, we also obtain $\phi(A^{-1}) = \phi(A)^{-1}$.

For the main result of this paper, we will characterize the anti-G-Hermiticity preserving continuous linear map $\tau : C(H) \rightarrow C(H)$ that preserves strongly the invertibility of the elements of $C(H)$.

Theorem 6.7. *Let $\phi : B(H) \rightarrow B(H)$ be an essentially anti-G-Hermiticity preserving continuous linear map which is unital and surjective up to compact operators. Then, the linear map τ induced by ϕ preserves anti-G-Hermiticity and preserves strongly the invertibility of the elements of $C(H)$ if for all A, $B \in B(H)$ such that $AB - BA$, $AB - I \in K(H)$, ϕ*

is a homomorphism up to compact operators. Further, if ϕ preserves strongly invertibility, then there exists $T \in B(H)^{-1}$ such that for all $A + K(H) \in C(H)$, $\tau (A + K(H)) = TAT^{-1} + K(H)$ or $\tau (A + K(H)) = TA^tT^{-1} + K(H)$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal basis of H.

Proof. The map τ induced by ϕ is an anti-G-Hermiticity preserving linear map that preserves strongly invertibility since by Lemma 5.2, τ preserves anti-G-Hermitian elements of $C(H)$ and by Proposition 6.2, τ preserves strongly the invertibility of the elements of $C(H)$.

Let $A + K(H) \in C(H)$. Then $A \in B(H)$. Since ϕ preserves strongly invertibility, by Proposition 6.6, there exists $T \in B(H)^{-1}$ such that $\phi (A) = TAT^{-1}$ or $\phi (A) = TA^tT^{-1}$. Thus, if $\phi (A) = TAT^{-1}$, then $\tau (A + K(H)) = \rho\phi (A) = TAT^{-1} + K(H)$, or if $\phi (A) = TA^tT^{-1}$, we get $\tau(A + K(H)) = TAT^{-1} + K(H)$.

Corollary 6.8. *Let $\phi : B(H) \to B(H)$ be an essentially anti-G-Hermiticity preserving linear map which is unital and surjective up to compact operators. If ϕ preserves strongly invertibility, then τ is a$^{\#}$-automorphism or a$^{\#}$-anti-automorphism.*

Proof. Let $A_1 + K(H)$, $A_2 + K(H) \in C(H)$. So, $\tau [(A_1 + K(H))(A_2 + K(H))] = \rho\phi (A_1A_2) = \phi (A_1A_2) + K(H)$. By Theorem 6.7, there exists $T \in K(H)^{-1}$ such that $\tau (A + K(H)) = TAT^{-1} + K(H)$ or $\tau (A + K(H)) = TA^tT^{-1} + K(H)$ for all $A + K(H) \in C(H)$. If $\tau (A + K(H)) = TAT^{-1} + K(H)$, then $\phi (A_1A_2) + K(H) = TA_1A_2T^{-1} + K(H) = (TA_1T^{-1} + K(H))(TA_2T^{-1} + K(H)) = \tau (A_1 + K(H))\tau (A_2 +K(H))$ so that $\tau [(A_1 + K(H))(A_2 + K(H))] = \tau(A_1 + K(H)) \tau (A_2 + K(H))$. Similarly, if $\tau (A + K(H)) = TA^tT^{-1} + K(H)$, then we get $\tau [(A_1 + K(H))(A_2 + K(H))] = \tau (A_2 + K(H)) \tau (A_1 + K(H))$. Hence, τ is a homomorphism or an anti-homomorphism.

Let $B + K(H) \in C(H)$. Then if $\tau (A + K(H)) = TAT^{-1} + K(H)$ for all $A + K(H) \in C(H)$, there exists $T^{-1}BT + K(H) \in C(H)$ such that $\tau (T^{-1}BT + K(H)) = B + K(H)$. If $\tau (A + K(H)) = TA^tT^{-1} + K(H)$ for all $A + K(H) \in C(H)$, then there exists $T^tB^t(T^{-1})^t + K(H) \in C(H)$ such that $\tau (T^tB^t(T^{-1})^t + K(H)) = B + K(H)$. Thus, τ is surjective.

Now, let us consider the case when $\tau (A + K(H)) = TAT^{-1} + K(H)$ for every $A + K(H) \in C(H)$. We let $B + K(H) \in C(H)$ such that $\tau (B + K(H)) = K(H)$. Then $TBT^{-1} + K(H) = K(H)$, which implies that $TBT^{-1} \in K(H)$. Since $K(H)$ has the absorbing property, $T^{-1}TBT^{-1}T = B \in K(H)$. Thus, $B + K(H) = K(H)$ so that Kerτ = $\{K(H)\}$. If $\tau (A + K(H)) = TA^tT^{-1} +K(H)$ for every $A + K(H) \in C(H)$, then it can also be shown similarly that Kerτ = $\{K(H)\}$. Hence, τ is injective.

By Lemma 5.2, τ preserves the G-adjoint of the elements of $C(H)$. Therefore, τ is a $^{\#}$-automorphism or a $^{\#}$-anti-automorphism.

SUMMARY AND RECOMMENDATION

This paper is devoted to the investigation of some linear preserving properties of some linear preservers on $C(H)$, particularly dealing with anti-G-Hermiticity and invertibility properties. We examined the preserving properties of the canonical map, the linear map that preserves essentially anti-G-Hermitian operators on H, and the map induced by the essentially anti-G-Hermiticity preserving linear map in relation to the linear map that preserves strongly the invertibility of operators on H. For an essentially anti-G-Hermiticity preserving linear map $\phi : B(H) \to B(H)$, which is unital and surjective up to compact operators, and the canonical map $\rho : B(H) \to C(H)$, we characterize the unique linear map τ induced by ϕ, such that $\tau\rho = \rho\phi$, under some assumptions on ϕ, with the inclusion that if ϕ preserves strongly invertibility, where ϕ has either the form $\phi (\cdot) = T(\cdot)T^{-1}$ or $\phi (\cdot) = T(\cdot)^tT^{-1}$ for some fixed T, and using this form of ϕ, the characterization of the induced map τ is made accordingly.

It can be observed however that the induced map, which is a mapping from $C(H)$ onto itself, is dependent on the linear maps $\phi : B(H) \rightarrow B(H)$ and $\rho : B(H) \rightarrow C(H)$. It would be interesting to look at the center $Z(C(H))$ of $C(H)$ and investigate the anti-G-Hermiticity and invertibility preserving properties of an independent map from $C(H)$ into itself and make a characterization of that linear map. It is also noteworthy to study other linear preservers on $C(H)$ and examine the G-properties that they preserve such as being G-unitary, G-quasi-unitary, and G-projections, among others.

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