Zero Ring Index of Cactus Graphs

Michelle Dela Rosa-Reynera\textsuperscript{1*} and Leonor Aquino-Ruivivar\textsuperscript{2}

\textsuperscript{1}Mathematics Department, Mariano Marcos State University, Quiling Sur, Batac City, Philippines
\textsuperscript{2}Mathematics and Statistics Department, De La Salle University, Manila, Philippines

*Corresponding Author: reyneramichelle@gmail.com; michelle_reynera@dlsu.edu.ph

ABSTRACT

A new notion of graph labeling called zero ring labeling is realized by assigning distinct elements of a zero ring to the vertices of the graph such that the sum of the labels of adjacent vertices is not equal to the additive identity of the zero ring. The zero ring index of a graph \(G\) is the smallest positive integer \(\xi(G)\) such that there exists a zero ring of order \(\xi(G)\) for which \(G\) admits a zero ring labeling. Any zero ring labeling of \(G\) is optimal if it uses a zero ring consisting of \(\xi(G)\) elements. It is known that any tree of order \(n\) has a zero ring index equal to \(n\). Considering that cactus graphs are interesting generalizations of trees, in this paper, we extend the optimal zero ring labeling scheme for trees to cactus graphs that leads us to establish that cactus graphs have also zero ring indices equal to their orders. The labeling was done using the zero ring \(M_2^0(Z_n)\).

Keywords: zero ring, zero ring labeling, zero ring index, cactus graph, spanning tree

INTRODUCTION

The concept of the zero ring index of graphs is associated with a notion of vertex labeling for graphs called zero ring labeling, which was introduced by Acharya et al. (2015). This new labeling technique is a topic that links graph theory with abstract algebra.

A ring \(R\) in which the product of any two elements is 0, where 0 is the additive identity of \(R\), is called a zero ring and is denoted by \(R^0\). Let \(G = (V, E)\) be a graph with vertex set \(V = : V(G)\) and edge set \(E = : E(G)\), and let \(R^0\) be a finite zero ring. An injective function \(f : V(G) \rightarrow R^0\) is called a zero ring labeling of \(G\) if \(f (u) + f (v) \neq 0\) for every edge \(uv \in E(G)\). The zero ring index of \(G\) is the smallest positive integer \(\xi(G)\) such that there exists a zero ring \(R^0\) of order \(\xi(G)\) for which \(G\) admits a zero ring labeling. Any zero ring labeling \(f : V(G) \rightarrow R^0\)
of $G$ is said to be optimal if $|R^0| = \xi(G)$. If $|V(G)| = n$ and $k = \lfloor\log_2 n\rfloor$, then it is known that $n \leq \xi(G) \leq 2^k$ (Acharya et al., 2015).

Several classes of graphs were found to have zero ring indices attaining the lower bound. These include the following: cycle graphs and the Petersen graph (Pranjali et al., 2014); complete graphs of order $2^k$ for some positive integer $k_0$ (Acharya et al., 2015); and fans, wheels, helms, gears, and friendship graphs (Reynera & Ruivivar, 2018). Moreover, Reynera and Ruivivar (2017) proved that bipartite graphs, a class of graphs that include trees, have zero ring indices equal to their orders. In that paper, a scheme in obtaining an optimal zero ring labeling for any tree of order $n$ using the zero ring $M_2^0(\mathbb{Z}_n) = \{[0 \ 0], [1 \ -1], \ldots, [n - 1 \ -(n - 1)]\}$ is presented. For a background on zero rings and related results, we refer to Pranjali and Acharya (2014).

The focus of this paper is to establish that cactus graphs also belong to the family of graphs with zero ring indices equal to their orders. This was done by extending the optimal zero ring labeling scheme for trees to cactus graphs.

**PRELIMINARIES**

For terminology and notation in graph theory not defined here, we refer to Chartrand and Zhang (2005).

A connected graph without any cycle is called a tree. A rooted tree is a tree in which a particular vertex is designated as the root and every edge is directed away from the root. The level of a vertex $v$ is the distance from the root to $v$. The height of a rooted tree is the maximum level of its vertices. In general, any tree $T$ can be redrawn as a rooted tree by designating any one vertex of $T$ as the root.

Given a tree $T$ of order $n$, an optimal zero ring labeling of $T$ using $M_2^0(\mathbb{Z}_n)$ can be obtained by applying the following algorithm, which is based on the paper by Reynera and Ruivivar (2017):

**Step 1.** Redraw $T$ as a rooted tree by designating any one vertex as the root.

**Step 2.** Partition the vertices of $T$ into partite sets $V_1$ and $V_2$ as follows:

All vertices of level $k$ in $T$, where $k$ is even, are placed in one set. This includes the root. The vertices of level $l$, where $l$ is odd, form another set.

The set with lesser cardinality will be $V_1$, and the other set will be $V_2$. If the two sets contain the same number of vertices, then either set can be chosen as $V_1$.

**Step 3.** Determine the sets of labels $L_1$ and $L_2$ for the vertices in $V_1$ and $V_2$, respectively.

If $A_i = \left[\begin{array}{cc} l & -i \\ i & l \end{array}\right] \in M_2^0(\mathbb{Z}_n)$, then we identify the elements of $L_1$ and $L_2$ as follows:

i. If $|V_1| = 1$, then $L_1 = \{ A_0 \}$ and $L_2 = \{ A_i \mid i = 1, 2, \ldots, n - 1 \}$.

ii. If $|V_1| = m$ is even, then $L_1 = \{ A_i \mid i = 1, 2, \ldots, \frac{m}{2}, n - 1, n - 2, \ldots, n - \frac{m}{2} \}$ and $L_2 = \{ A_i \mid i = 0, \frac{m}{2} + 1, \frac{m}{2} + 2, \ldots, n - 1 - \frac{m}{2} \}$.

iii. If $|V_1| = m$ is an odd integer greater than 1, then $L_1 = \{ A_i \mid i = 0, 1, 2, \ldots, \frac{m - 1}{2}, n - 1, n - 2, \ldots, n - \frac{m - 1}{2} \}$ and $L_2 = \{ A_i \mid i = \frac{m + 1}{2}, \frac{m + 1}{2} + 1, \ldots, n - 1 - \frac{m - 1}{2} \}$.

**Step 4.** Assign arbitrarily the elements of $L_1$ and $L_2$ as distinct labels of the vertices in $V_1$ and $V_2$, respectively.

It can be observed that for every pair of elements $A_i, A_{n-i} \in M_2^0(\mathbb{Z}_n)$ such that $A_i \neq A_{n-i}$,
we have either \( A_i, A_{n-i} \in L_1 \) or \( A_i, A_{n-i} \in L_2 \). In other words, the pairs of distinct elements in \( M^0_2(\mathbb{Z}_n) \) that are negatives of each other are labels of vertices belonging to one partite set.

A **cactus graph** is a connected graph in which any two cycles have at most one vertex in common. Equivalently, any edge of a cactus graph lies on at most one cycle.

A **spanning tree** of a graph \( G \) is a subgraph that is a tree that includes all the vertices of \( G \). It is well known that every connected graph has a spanning tree. Hence, every cactus graph has a spanning tree. An edge in a spanning tree \( T \) is called a **branch** of \( T \). An edge of \( G \) that is not in a given spanning tree \( T \) is called a **chord**.

A cactus graph of order 11 is shown in Figure 1, where the bold lines represent the branches of a spanning tree. The dashed lines are the chords of the cactus graph with respect to the given spanning tree.

![Figure 1. A spanning tree of a cactus graph.](image)

We note that branches and chords are defined with respect to a given spanning tree. An edge that is a branch of one spanning tree \( T_1 \) in a cactus graph \( G \) may be a chord with respect to another spanning tree \( T_2 \).

The fact that every cactus graph has a spanning tree motivated us to extend the optimal zero ring labeling scheme for trees to cactus graphs.

**RESULTS AND DISCUSSION**

We start with some results that are useful in establishing that any cactus graph has a zero ring index equal to its order.

**Lemma 1.** In every cactus graph \( G \) containing at least two cycles, there exists a spanning tree such that any two chords of \( G \) are not incident with a common vertex.

*Proof:* Consider a cactus graph \( G \) that contains at least two cycles. By definition, any two cycles in \( G \) have at most one common vertex.

If the cycles in \( G \) are pairwise vertex disjoint, then the required spanning tree is obtained by deleting any edge in each cycle.

If \( C^1 \) and \( C^2 \) are two cycles in \( G \) having vertex \( v \) as their common vertex, delete an edge in \( C^1 \) that is incident with \( v \). Since any cycle has at least three edges and \( C^2 \) has only two edges incident with \( v \), there is an edge \( e \) of \( C^2 \), which is not incident with \( v \). We delete \( e \) in \( C^2 \). On the other hand, if a cycle in \( G \) has no vertex in common with another cycle, we can delete any one edge in this cycle. The resulting graph is a spanning tree of \( G \). Moreover, with respect to this spanning tree, any two chords of \( G \) are not incident with a common vertex.

The next result describes the absolute difference between the levels of the end vertices of a chord of a cycle when its spanning tree is redrawn as a rooted tree.

**Lemma 2.** Let \( e = uv \) be an edge of the cycle \( C_n \). Suppose \( C_n - e \) is redrawn as a rooted tree by designating any one vertex as the root such that \( s \) and \( t \) denote the levels of the vertices \( u \) and \( v \), respectively. Then,

i. \( |s - t| \) is odd when \( n \) is even; and
ii. \( |s - t| \) is even when \( n \) is odd.

*Proof:* Consider the cycle \( C_n = [v_0, v_1, \ldots, v_{n-1}, v_0] \). Without loss of generality, let \( e = v_0v_{n-1} \). Then, \( C_n - e \) is the path graph
[\nu_0, \nu_1, \ldots, \nu_{n-1}]$. Choosing \(\nu_i\) as a root, for some \(i = 0, 1, \ldots, n-1\), we obtain a rooted tree of height equal to \(\max\{i, n-1-i\}\). In the rooted tree, note that \(\nu_0\) is of level \(s = i\) and \(\nu_{n-1}\) is of level \(t = n-1-i\). Thus, \(|s-t| = |i-(n-1-i)| = |2i-n+1|\), which is odd when \(n\) is even and is even when \(n\) is odd.

The following result shows the possibility of obtaining \(k\) distinct pairs of elements that are not negatives of each other from a subset of a zero ring.

**Lemma 3.** Let \(R^0\) be a zero ring of order \(n\), where 0 is the additive identity. Suppose \(x_i, y, z \in R^0\) such that \(x_i \neq -x_i\) and the sets \({x_i, -x_i}\) for all \(1 \leq i \leq k\) are pairwise disjoint. Consider the following subsets of \(R^0\):

i. \(A = \{x_1, x_2, \ldots, x_k, -x_1, -x_2, \ldots, -x_k\}\), where \(2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor\) and \(n \geq 5\);

ii. \(B = \{y, x_1, x_2, \ldots, x_k, -x_1, -x_2, \ldots, -x_k\}\), where \(1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor\), \(n \geq 3\) and either \(y = -y\) or \(-y \in B\); and

iii. \(C = \{0, z, x_1, x_2, \ldots, x_{k-1}, -x_1, -x_2, \ldots, -x_{k-1}\}\), where \(1 \leq k-1 \leq \left\lfloor \frac{n-1}{2} \right\rfloor\), \(n \geq 4\) and either \(z = -z\) or \(-z \in C\).

Then, in each of these subsets, we can form \(k\) pairwise disjoint sets of two elements that are not negatives of each other.

**Proof:** In \(A\), consider the pairs \({x_i, -x_{i+1}}\) for \(1 \leq i \leq k-1\) and \({x_k, -x_1}\). We have formed \(k\) pairwise disjoint sets of two elements that are not negatives of each other.

In \(B\), we have \({y, x_1}\) when \(k = 1\). If \(k \geq 2\), then \(B = A \cup \{y\}\). Thus, the pairs of elements identified in \(A\) can also be considered in \(B\).

Lastly, in \(C\), we take the pairs \({0, x_1}\) and \({z, -x_1}\) when \(k = 2\). If \(k \geq 3\), then \({0, z}\), \({x_k-1, -x_1}\), and \({x_i, -x_{i+1}}\) for \(1 \leq i \leq k-2\) are \(k\) pairwise disjoint sets of two elements that are not negatives of each other.

We now prove the main theorem. It is important to note that among all cactus graphs of order \(n \leq 3\), only the cycle \(C_3\) has zero ring index not equal to its order.

**Theorem 4.** For any cactus graph \(G\) of order \(n > 3\), \(\xi(G) = n\).

**Proof:** Let \(G\) be a cactus graph of order \(n\). We consider two cases for \(G\).

**Case 1.** If \(G\) has no odd cycles, then \(G\) is bipartite. It follows from the result of Reynera and Ruivivar (2017) that \(\xi(G) = n\). In this case, an optimal zero ring labeling of a spanning tree \(T\) of \(G\) using \(M_2^0(\mathbb{Z}_n)\) is also a zero ring labeling of \(G\). This is because a chord coming from an even cycle of \(G\) joins a vertex in \(V_1\) with a vertex in \(V_2\) as implied by Lemma 2.

**Case 2.** If \(G\) contains the odd cycles \(C_1, C_2, \ldots, C_t\), where \(t \geq 1\), obtain a spanning tree \(T\) of \(G\) by removing one edge, say \(e_j = u_jv_j\), from each \(C_j\), \(1 \leq j \leq t\), such that the sets \({u_j, v_j}\) are pairwise disjoint. This is possible by Lemma 1.

Our aim is to extend the optimal zero ring labeling scheme for \(T\) to the cactus graph \(G\) with odd cycles.

Perform Steps 1 to 3 in the given labeling scheme for trees to the spanning tree \(T\) of \(G\) to determine the sets of labels \(L_1\) and \(L_2\) for the vertices in the partite sets \(V_1\) and \(V_2\), respectively, of the vertex set of \(T\).

By the way the vertices of \(T\) are classified into \(V_1\) and \(V_2\) in Step 2, it follows from Lemma 2 that the difference of the levels of \(u_j\) and \(v_j\), \(1 \leq j \leq t\), in the corresponding rooted tree of \(T\) is even. This means that either \(u_j, v_j \in V_1\) or \(u_j, v_j \in V_2\). Thus, an arbitrary assignment of labels from \(L_1\) (or \(L_2\)) to the vertices in \(V_1\) (or \(V_2\)) may lead to the labels of \(u_j\) and \(v_j\) being negatives of each other. To avoid this, we modify Step 4 in the labeling scheme for trees as generalized to a cactus graph \(G\) with odd cycles.
In assigning labels to the vertices of $G$, we consider three subcases.

**Subcase 2.1.** Suppose $|V_1| = 1$ and $|V_2| \geq 3$. Then, $L_1 = \{A_0\}$ and $L_2 = \{A_1, A_2, ..., A_{n-1}\}$. Here, the rooted tree corresponding to the spanning tree $T$ of $G$ has a height of 1, and hence, all cycles in $G$ are of length 3. The lone vertex in $V_1$ is labeled as $A_0$. Using Lemma 3, every pair of vertices in $V_2$ that forms a chord can be labeled by two elements of $L_2$ whose sum is not equal to $A_0$. The unused elements of $L_2$ are used to label the remaining vertices in $V_2$.

**Subcase 2.2.** Suppose $|V_1| \geq 2$ and the vertices in $V_1$ do not form a chord from an odd cycle when $|V_1| = 2$.

If $|V_1| = |V_2| = 2$, then $L_1 = \{A_1, A_3\}$ and $L_2 = \{A_0, A_2\}$. Moreover, if $|V_1| = 2$ and $|V_2| \geq 3$, then $L_1 = \{A_1, A_{n-1}\}$ and $L_2 = \{A_0, A_2, A_3, ..., A_{n-2}\}$. Even though the sum of the labels of the vertices in $V_1$ is $A_0$, they are nonadjacent vertices in $G$ because they do not form a chord from an odd cycle. In assigning labels to the vertices of $G$ that are contained in partite sets with at least three elements, we first assign elements of $L_1$ (or $L_2$) that are not negatives of each other as labels of the vertices $u_j, v_j \in V_1$ (or $V_2$) of a chord from an odd cycle in $G$. This is possible by Lemma 3. The unused elements of $L_1$ (or $L_2$) may then be arbitrarily assigned as labels to the remaining vertices in $V_1$ (or $V_2$).

**Subcase 2.3.** Suppose $|V_1| = 2$, $|V_2| \geq 3$ and the vertices in $V_1$ form a chord from an odd cycle. Without loss of generality, let $u_1, v_1 \in V_1$.

From Step 3, we have $L_1 = \{A_1, A_{n-1}\}$ and $L_2 = \{A_0, A_2, A_3, ..., A_{n-2}\}$. In this case, the sum of the labels of $u_1$ and $v_1$ would be $A_0$. So, for this case, Step 3 must also be modified, in addition to Step 4. That is, we modify $L_1$ and $L_2$ by interchanging $A_{n-1}$ with $A_0$. We thus obtain $L'_1 = \{A_0, A_1\}$ and $L'_2 = \{A_2, A_3, ..., A_{n-1}\}$ as new sets of labels for the vertices in $V_1$ and $V_2$, respectively.

Since $G$ has at least five vertices and $|V_1| = 2$, the rooted tree $T_r$ that corresponds to the spanning tree $T$ of $G$ has a height of 2, 3, or 4. Let us specifically identify the vertices $u_1, v_1 \in V_1$ in the different forms of $T_r$.

i. $T_r$ has a height of 2 such that only $u_1$ and $v_1$ are vertices of level 1 and satisfies exactly one of the following:
   a. There is no vertex of level 2 adjacent to $v_1$ and at least two vertices of level 2 are adjacent to $u_1$.
   b. Each $u_1$ and $v_1$ is adjacent to at least one vertex of level 2.

ii. $T_r$ has a height of 3 with $u_1$ designated as the root, $v_1$ is the only vertex of level 2, there is at least one vertex of level 3 adjacent to $v_1$, and there are at least two vertices of level 1 adjacent to $u_1$.

iv. $T_r$ has a height of 3 with $v_1$ designated as the root, $u_1$ is the only vertex of level 2, there is only one vertex of level 1 adjacent to $v_1$, and there are at least two vertices of level 3 adjacent to $u_1$.

v. $T_r$ has a height of 4 with only $u_1$ at level 1, only $v_1$ is at level 3, there is at least one vertex of level 2 adjacent to $u_1$, and there is at least one vertex of level 4 adjacent to $v_1$.

In all the above nonisomorphic forms of $T_r$, we label $u_1$ by $A_0$ and $v_1$ by $A_1$. In this way, the sum of the labels of the vertices $u_1$ and $v_1$ is not equal to $A_0$.

Next, we find a vertex $x \in V_2$ in which we could assign the label $A_{n-1}$ such that $v_1x$ is neither a branch of $T$ nor a chord from an even cycle of $G$. Otherwise, the sum of the labels of

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$v_1$ and $x$ may possibly be $A_1 + A_{n-1} = A_0$. We will show that this can be done in each of the five cases from (i) to (v).

In (i.a) and (i.b), we choose $x$ to be any one vertex of level 2 that is adjacent to $u_i$. In (ii) and (v), we take $x$ to be the root of $T_r$. In (iii), note that there are at least two vertices of level 1 that are adjacent to the root $u_1$. Since $v_1$, which is of level 2, can be adjacent to only one vertex of level 1, it follows that $x$ can be chosen to be any vertex of level 1 that is not adjacent to $v_1$. Lastly, in (iv), $x$ is chosen to be any vertex of level 1 that is not adjacent to $v_1$. We thus modify Step 4 in the labeling scheme to label vertices coming from odd cycles that are incident with vertices in $V_2$, we note that we can pick two elements of $L_2^* \{ A_{n-1} \}$ that are not negatives of each other as labels to vertices $u_j$ and $v_j$, $2 \leq j \leq t$. If there are still unlabelled vertices in $V_2$, then we can arbitrarily use the remaining elements of $L_2^*$ to label these vertices.

The resulting labeling is an optimal zero ring labeling of any cactus graph $G$ of order $n > 3$, since $|M_2^*(Z_n)| = n$. Therefore, $\xi(G) = n$.

**CONCLUSION**

In this study, we have investigated the zero ring index of cactus graphs. It is shown that all cactus graphs of order $n > 3$ belong to the family of graphs with zero ring indices equal to their orders. This is done by extending the optimal zero ring labeling scheme for trees to cactus graphs using the zero ring $M_2^*(Z_n)$. In future studies, one may construct optimal zero ring labelings of different classes of graphs using other zero rings. It would also be interesting to characterize graphs with zero ring indices attaining the lower bound in addition to what Pranjali et al. (2014) have established.

![Figure 2. Nonisomorphic forms of $T_r$.](image-url)
REFERENCES


