Generating Spanning Maximal Planar Subgraphs of Complete 4-Partite Graphs

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ABSTRACT

A *spanning maximal planar subgraph (SMPS)* T of a simple, finite, undirected graph G is a spanning subgraph of G that is also a maximal planar graph. In this paper, we introduce some methods of constructing complete 4-partite graphs $K_{w,x,y,z}$ with SMPS. We utilize these methods to the SMPS problem for complete tripartite graphs to generate complete 4-partite graphs with SMPS and provide some relationships between the cardinalities of the two graphs.

Keywords: spanning maximal planar subgraph, complete 4-partite graph

INTRODUCTION

A graph is said to be a *planar graph* if there is a drawing of the graph on the plane such that there are no edge crossings. Otherwise, the graph is *non-planar*. For a given graph $G=(V,E)$, the aim of a graph planarization problem is to seek a subset $F \subseteq E$ with minimum cardinality such that the subgraph of H of G with edge set $E \backslash F$ is a planar graph. In other words, it is required to remove a minimum number of edges from G and obtain a planar graph. Such a problem belongs to the class of NP-hard problems (Liu & Geldmacher, 1977). This means that it is difficult to develop an algorithm efficient enough to solve the problem as the number of vertices of G increases. A special planar graph with the property that the addition of any edge joining two non-adjacent vertices results in a non-planar graph is called a *maximal planar graph*. A related problem to the graph planarization problem is to determine if it is possible to remove a minimum number of edges from G, resulting to a maximal planar graph T. Necessarily, the graph T is a spanning subgraph of G since none of the vertices were removed. It is because of these characteristics that we will refer to the graph

T as a *spanning maximal planar subgraph* or an "SMPS" of G, and finding this subgraph, if it exists, will be referred to as the SMPS problem. The SMPS problem was tackled by Gervacio et al. (2017) for complete tripartite graphs, where such graphs with an order greater than 6 were considered and it was identified which of these graphs have an SMPS. Now consider the complete 4-partite graph $K_{2,2,2,2}$. Figure 1.1 shows the existence of an SMPS for $K_{2,2,2,2}$, by deleting six edges and drawing the obtained graph such that all regions are emphasized to be triangular regions. The labels 1,2,3, and 4 refer to a vertex's membership to the partite sets V_1, V_2, V_3, and V_4, respectively.

In this paper, we discuss some results to the SMPS problem for complete 4-partite graphs. Some methods of generating larger SMPS for the case of complete 4-partite graphs are presented here. These methods are applied to complete tripartite graphs $K_{x,y,z}$ with SMPS, resulting to SMPS of complete 4-partite graphs $K_{w',x',y,z'}$.

MATERIALS AND METHODS

We discuss in this section the required concepts in graph planarity to go through the outputs of this paper. These concepts include some properties of general planar graphs and maximal planar graphs. All graphs throughout the text are generated with the aid of GraphTeX 2.0 (Gervacio, 2008). We first define the graph under study formally, the complete k-partite graph.

Definition 2.1. A k-partite graph $G=(V,E)$ is any graph with the characteristic that V may be partitioned into k non-empty subsets $V_1,V_2,\ldots,V_k$ such that there exists no edge $uv\in E$ for which $u,v\in V_i$, where $1\le i\le k$. If for every vertex $u\in V_i$ and $v\in V_j$, $1\le i<j\le k$, there exists an edge $uv\in E$, then G is called a complete-partite graph.

The usual notation for a complete k-partite graph is $K_{t_1,t_2,\ldots,t_k}$, where $|V_i|=t_i$. The subscripts $t_1,t_2,\ldots,t_k$ may also be arranged in ascending order, without loss of generality. This is assumed for convenience whenever the structure of this graph is studied. For k = 2 and k = 3, there are special names: complete bipartite and complete tripartite, respectively. The standard method of drawing k-partite graphs is by grouping the vertices according to their membership to the partite sets. In the discussions, we will use the integers 1,2,3, and 4 to denote the membership of a vertex to the partite sets V_1, V_2, V_3, and V_4, respectively.

When a planar graph is drawn on the plane without any edge-crossings, the plane is divided into non-overlapping, open regions. This includes an infinite exterior region.

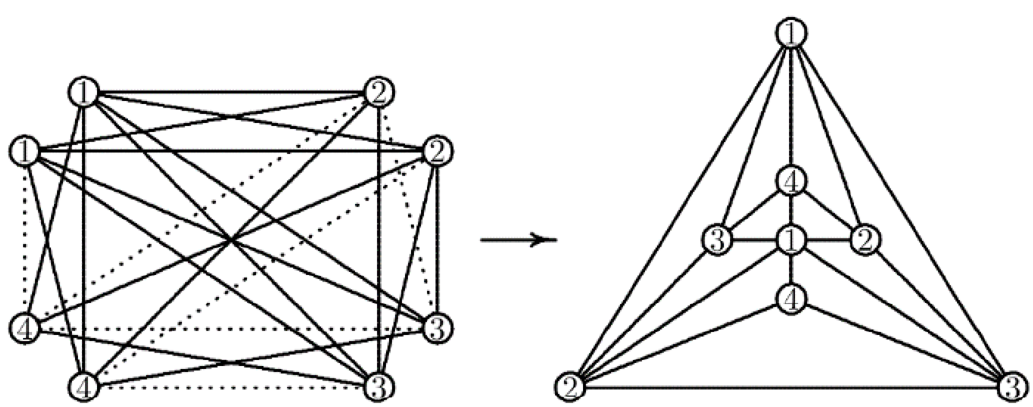

Figure 1.1. An SMPS of $K_{2,2,2,2}$

A well-known property of a planar graph was presented by Euler (1754), and his proof was later corrected by Legendre (1794). It states the relationship between the order, size, and the number of regions of any planar graph:

Theorem 2.2. If G is a connected planar graph with order n and size m and contains r regions, then $n - m + r = 2$.

If a planar graph G has the property that a non-planar graph G' is obtained from adding an edge to join two non-adjacent vertices of G, then G is referred to as a *maximal planar graph*. Maximal planar graphs can be drawn in such a way that the exterior region is bounded by a large triangle. Any region of a planar graph, in fact, can be the exterior region. This becomes evident if the graph is drawn on the sphere. Two drawings of a maximal planar graph are shown in the next figure. Figure 2.1 (b) will be the desired way of drawing a maximal planar graph, as it emphasizes the property that each region is triangular.

The statement below is a corollary to *Theorem 2.2*. It relates the number of edges and number of vertices in a maximal planar graph. It will be useful in establishing relationships between the cardinalities of complete tripartite graphs and complete 4-partite graphs with SMPS:

Corollary 2.3. If G is a maximal planar graph with order n and size m, then $m = 3n - 6$.

A *polyhedron* is a 3-dimensional object whose boundary is a set of polygonal plane surfaces. Among the well-known polyhedra are the so-called *five platonic solids*: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. We will be interested only with the platonic solids whose boundary is composed of triangular surfaces—the tetrahedron, octahedron, and icosahedron. When each of these polyhedra is projected onto a plane, a map equivalent to a maximal planar graph is produced. The projections of these polyhedra are shown in Figure 2.2.

It is clear that the projections of a tetrahedron, octahedron, and icosahedron onto the plane are maximal planar graphs. These

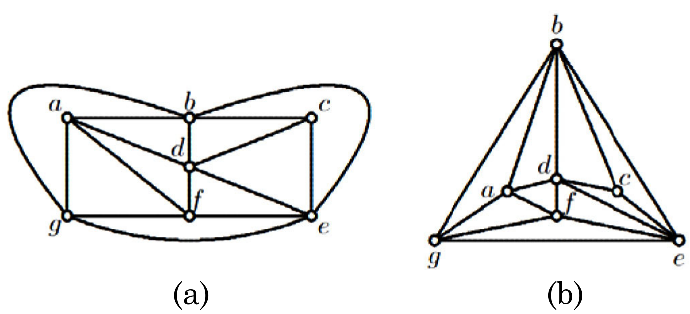

Figure 2.1. Two drawings of a maximal planar graph.

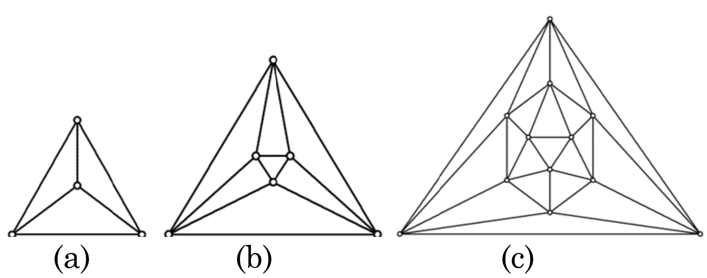

Figure 2.2. A (a) tetrahedron, (b)octahedron, and (c)icosahedron projected onto a plane.

graphs may be utilized to construct larger maximal planar graphs and will be discussed in the main sections.

RESULTS AND DISCUSSION

Generating Spanning Maximal Planar Subgraphs

This section discusses a few methods of constructing larger SMPS, given an SMPS of some complete 4-partite graph $K_{w,x,y,z}$. In particular, these methods discuss how to add vertices and consequently add edges to preserve maximal planarity and obtain an SMPS of a complete 4-partite graph with higher order than the given SMPS. In each result, we show one case only, as the rest of the cases are treated in an analogous manner using the same modification process.

The first result deals with modifying an edge of an SMPS of a complete 4-partite graph by inserting two vertices. This modification was introduced in the paper of Gervacio et al. (2017) for complete tripartite graphs.

Proposition 3.1. If $K_{w,x,y,z}$ contains an SMPS, then each of $K_{w+1,x,y,z+1}$, $K_{w+1,x,y+1,z}$, $K_{w+1,x+1,y,z}$, $K_{w,x+1,y,z+1}$, $K_{w,x+1,y+1,z}$, and $K_{w,x,y+1,z+1}$ contains an SMPS.

Proof. Suppose that T is an SMPS of $K_{w,x,y,z}$. Consider two adjacent regions of T induced by the sets $\{a, b, c\}$ and $\{a, b, d\}$ with ab as the common edge. We show one case: if $a \in V_1$ and $b \in V_2$. Hence, $c, d \in V_3 \cup V_4$. Modify T into a new graph T' by inserting new vertices $e \in V_2$ and $f \in V_1$ on the edge ab, thus deleting ab. Add the following edges to preserve maximal planarity: ae, ef, bf, ce, cf, de, df. Figure 3.1 illustrates how this modification of T, is carried out.

Clearly, T' is a maximal planar graph. Further, T' is an SMPS of $K_{w+1,x+1,y,z}$ since one vertex was added each to the partite sets V_1 and V_2. The other required graphs, $K_{w,x,y+1,z+1}$, $K_{w,x+1,y+1,z}$, and $K_{w,x,y+1,z+1}$, are shown to have an SMPS by considering the other possible inclusions of a and b to V_1, V_2, V_3, and V_4.

It was mentioned in the preliminaries that among the five platonic solids, the tetrahedron, octahedron, and icosahedron are in fact maximal planar graphs when they are projected onto a plane. The next results demonstrate how to modify a region by utilizing a tetrahedron, octahedron, or icosahedron and obtain an SMPS for a larger complete 4-partite graph.

Proposition 3.2. If $K_{w,x,y,z}$ contains an SMPS, then the following graphs also contain an SMPS: $K_{w+1,x,y,z}$, $K_{w,x+1,y,z}$, $K_{w,x,y+1,z}$, and $K_{w,x,y+1,z+1}$.

Proof. Consider a region of induced by a set of vertices $\{a, b, d\}$. We consider the case where $a \in V_2$, $b \in V_3$ and $c \in V_4$. Let T' be the new graph obtained by modifying R thru the addition

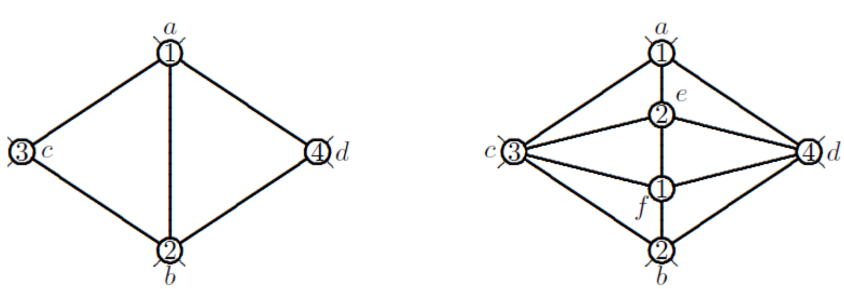

Figure 3.1. Two adjacent regions of T modified by adding two vertices to the common edge.

of a new vertex, $d \in V_1$, and attaching this new vertex to each of the boundary vertices a, b, and c. Clearly, T' is a maximal planar graph, since the triangular region R in T is modified into a tetrahedron in T'. The process is described in Figure 3.2.

Proposition 3.3. If $K_{w,x,y,z}$ contains an SMPS, then the following graphs contain an SMPS: $K_{w+1,x+1,y+1,z}$, $K_{w+1,x+1,y,z+1}$, $K_{w+1,x,y+1,z+1}$, and $K_{w,x+1,y+1,z+1}$.

Proof. Let T be an SMPS of a complete 4-partite graph $K_{w,x,y,z}$, and consider a region R induced by the set of vertices $\{a, b, c\}$ where $a \in V_1$, $b \in V_2$, and $c \in V_3$. Modify R in the maximal planar graph T into new graph T', by adding three vertices in its interior, say d, e, f, together with the following edges: ae, af, bd, bf, cd, ce, de, ef, and df. That is, modify R into an octahedron. Therefore, the new graph T' is a maximal planar graph since an octahedron is also a maximal planar graph. Further, since T' is a 4-partite graph, the vertices , and must each belong to a partite set. One such possible configuration is given by $d \in V_1$, $e \in V_2$, and $f \in V_3$. Hence, this configuration generates an SMPS T', of the complete 4-partite graph $K_{w+1,x+1,y+1,z}$. Refer to Figure 3.3 for an illustration of this modification.

The other possible configurations for the inclusions of d, e and f to V_1, V_2, V_3, and V_4 lead to the other required graphs with SMPS.In particular, the configurations (i) $d \in V_1, e \in V_2, f \in V_4$ (ii) $d \in V_4, e \in V_2, f \in V_3$ and (iii) $d \in V_1$, $e \in V_4, f \in V_3$ imply that $K_{w+1,x+1,y,z+1}$, $K_{w,x+1,y+1,z+1}$, and $K_{w+1,x,y+1,z+1}$, respectively, contain an SMPS. The other cases where the set $\{a, b, c\}$ has other configurations as to membership to the four partite sets are treated analogously, with the same modification process.

The following lemma is required in proving the next result. It is proven by first showing that for the nine interior vertices of an

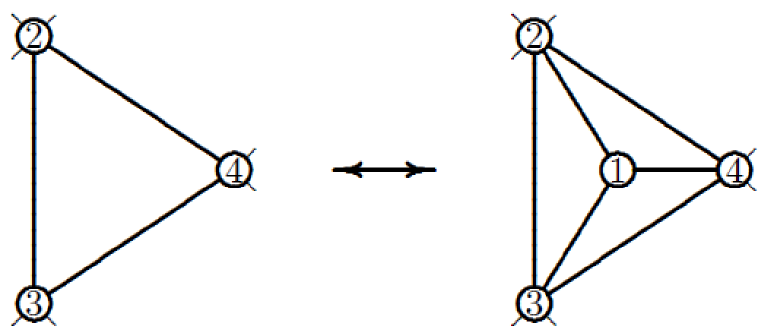

Figure 3.2. Modifying a region of T into a tetrahedron.

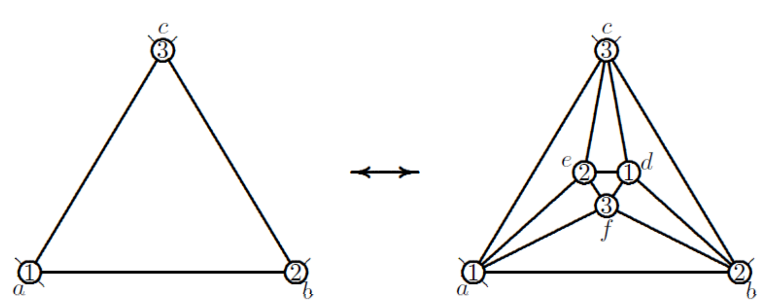

Figure 3.3. Modifying a region of T' into an octahedron.

icosahedron, the partite sets V_1, V_2, and V_3 can have at most two vertices each. Afterwards, it can be shown by contradiction that none among these three partite sets can have a cardinality of 1 when counting is restricted to the nine interior vertices. It can therefore be concluded that for the nine interior vertices, three must be in V_4.

Lemma 3.4. If G is an icosahedron whose exterior vertices belong to partite sets V_1, V_2, and V_3, then its nine interior vertices are partitioned such that two vertices are in each of V_1, V_2 and V_3 and three vertices are in the partite set V_4.

Proposition 3.5. If G contains an SMPS, then the following graphs contain an SMPS: $K_{w+2,x+2,y+2,z+3}$, $K_{w+2,x+2,y+3,z+2}$, $K_{w+2,x+3,y+2,z+2}$, and $K_{w+3,x+2,y+2,z+2}$.

Proof. Let $K_{w,x,y,z}$ be an SMPS of a complete 4-partite graph. We observe a region R in T that is induced by the vertices $a_1 \in V_1$, $a_2 \in V_2$ and $a_3 \in V_3$. Modify T into a new graph T' having the following vertex and edge sets:

$$V(T') = V(T) \cup \{b_1, c_1, b_2, c_2, b_3, c_3, a_4, b_4, c_4\} E(T')$$

$$E(T') = E(T) \cup \{a_1a_4, a_1b_2, a_1c_4, a_2c_4, a_2b_3, a_2b_4, a_3b_4, a_3b_1, a_3a_4, b_1b_4, b_4b_3, b_3c_4, c_4b_2, b_2a_4, a_4b_1, b_1c_2, b_4c_2, b_3c_2, b_3c_1, c_4c_1, b_2c_1, b_2c_3, a_4c_3, b_1c_3, c_3c_2, c_2c_1, c_1c_3\}$$

that is, add the vertices b_i, c_i, to V_i, where $1 \leq i \leq 3$, and add the vertices a_4, b_4, c_4, to V_4, then add the edges enumerated above to form an icosahedron. The figure below shows the resulting icosahedron after this modification.

Thus, T' is a maximal planar graph, since the triangular region R in T was modified into an icosahedron, which is a maximal planar graph. Furthermore, the nine interior vertices belonging to the set $\{b_1, c_1, b_2, c_2, b_3, c_3, b_4, c_4\}$ are partitioned such that two vertices are in each of V_1, V_2, and V_3 and three vertices are in V_4, from *Lemma 3.4*. Hence, it follows that the new graph T' is an SMPS of $K_{w+2,x+2,y+2,z+3}$.

The other required graphs $K_{w+2,x+2,y+3,z+2}$ $K_{w+2,x+3,y+2,z+2}$, and $K_{w+3,x+2,y+2,z+2}$ are all shown to have SMPS by considering the other possible configurations of the exterior vertices a_1, a_2, and a_3 as to their inclusions to the four partite sets and using an analogous argumentation.

Generating Larger SMPS From Complete Tripartite Graphs

The methods of adding vertices presented in the previous section could be utilized to produce other results for the SMPS problem of complete 4-partite graphs. In Gervacio et al. (2017), it was shown that there are complete tripartite graphs that contain SMPS. In particular, for complete tripartite graphs with order at most 9, the graphs $K_{1,1,1}$, $K_{2,2,2}$, $K_{2,3,3}$, and $K_{3,3,3}$ are the only ones having an

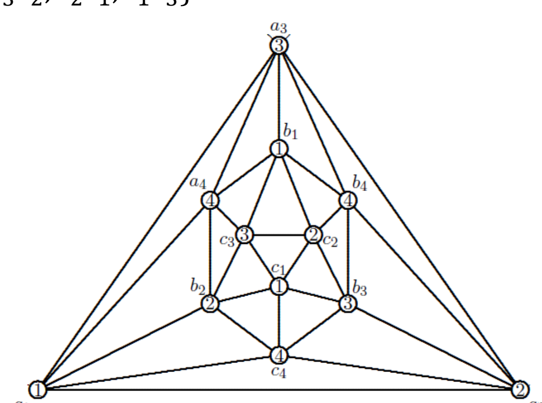

Figure 3.3. The resulting icosahedron in T'.

SMPS. In this section, we present a corollary stating that an SMPS of a complete 4-partite graph $K_{w,x,y,z}$ can also be constructed from an SMPS of a complete tripartite graph with the application of the previous results.

Corollary 4.1. Suppose that the complete tripartite graph contains an SMPS. Then, the following complete 4-partite graphs contain an SMPS:

(i) $K_{w,x,y,z}$
(ii) $K_{w+z_1,x+z_2,y+z_3,z_4}$
(iii) $K_{w+2z,x+2z,y+2z,3z}$

where $z_1 + z_2 + z_3 + z_4 = 3z$, $z_4 \geq 1$ and $z \leq 2w + 2x + 2y - 4$.

Proof. Suppose that $K_{w,x,y}$ is a complete tripartite graph with an SMPS T. Since T is a maximal planar graph, from Theorem 2.2, the equation $n - m + r = 2$ is satisfied by T, where n, m, and r are the order, size, and number, respectively, of regions in T. By Corollary 2.3, we have

$$r = m - n + 2 = (3n - 6) - n + 2 = 2n - 4$$

so that $r = 2w + 2x + 2y - 4$, since the order of T is $n = w + x + y$. This number of regions is considered for each of the graphs in (i), (ii), and (iii).

(i) Let $\{a, b, c\}$ be a set of vertices in T that induce a region. Since a,b and c are adjacent to one another, assume without loss of generality that $a \in V_1$, $b \in V_2$, and $c \in V_3$. Modify T into a new graph U with the following vertex and edge sets: $V(U) = V(T) \cup \{d\}$ and $E(U) = E(T) \cup \{ad,bd,cd\}$, where the new vertex d belongs to a new partite set V_4. Thus, U is a 4-partite graph. Further, it is a maximal planar graph since the triangular region induced by $\{a, b, c\}$ was modified into a tetrahedron. In particular, U is an SMPS of $K_{w,x,y,1}$. Now note that T contains $r = 2w + 2x + 2y - 4$ regions, and each of these regions are induced by vertices in V_1, V_2, and V_3. Hence, the process of adding a vertex to a region as in *Proposition 3.2* can be performed at most $2w + 2x + 2y - 4$ times. Therefore, an SMPS, say T' of the complete 4-partite graph $K_{w,x,y,z}$ can be generated from T if $z \leq 2w + 2x + 2y - 4$.

(ii) Consider a region induced by $\{v_1, v_2, v_3\}$ in T. Let the vertices a, b, and c be added to this region as in the process described in the proof of *Proposition 3.3*. That is, let U be the 4-partite graph obtained from T using the following vertex and edge sets:

$$V(U) = V(T) \cup \{a, b, c\}$$
$$E(U) = E(T) \cup \{av_1, bv_1, bv_2, cv_2, av_3, cv_3, ab, bc, ac\}$$

Since the vertices a, b, and c are adjacent to each other in U and U is a 4-partite graph, there is exactly one vertex in $\{a, b, c\}$, say a, belonging to a new partite set V_4. Thus, b and c belong to $V_1 \cup V_2$, $V_1 \cup V_3$, or $V_2 \cup V_3$. Hence, U is an SMPS of $K_{w+1,x+1,y,1}$, $K_{w+1,x,y+1,1}$, or $K_{w+1,x,y+1,1}$, respectively. Similar as in (i), this process of adding three vertices to a region may be performed at most $z = 2w + 2x + 2y - 4$ times. Moreover, since three vertices are added to the interior of each of the z regions, an SMPS T', of the complete 4-partite graph of the form $K_{w+z_1,x+z_2,y+z_3,z_4}$ may be generated from T where $z_1 + z_2 + z_3 + z_4 = 3z$, $z_4 \geq 1$, and $z \leq 2w + 2x + 2y - 4$.

(iii) We follow the process described in the proof of *Proposition 3.5*. Modify the SMPS T of$K_{w,x,y}$ into a 4-partite graph U, by adding nine vertices in the interior of a region induced by $\{a, b, c\}$ and adding the necessary edges to form an icosahedron whose vertices belong to partite sets V_1,

V_2, V_3, and V_4. Since a, b, and c are vertices in $V_1 \cup V_2 \cup V_3$, *Lemma 3.4* implies that two vertices were added to each of V_1, V_2, and V_3, while three vertices were added to V_4. Thus, U is an SMPS of the complete 4-partite graph $K_{w+2,x+2,y+2,3}$. Similar as in (i) and (ii), this process may be performed at most $z = 2w + 2x + 2y - 4$ times. Since there is only one possible configuration as to the inclusion of the nine vertices to V_1, V_2, V_3, and V_4 in the resulting z icosahedrons, the complete 4-partite graph $K_{w+2z,x+2z,y+2z,3z}$ contains an SMPS if $z \leq 2w + 2x + 2y - 4$.

CONCLUSION

In this paper, we have shown that given an SMPS of some complete 4-partite graph, a larger SMPS may be constructed for a complete 4-partite graph with higher order. These methods include inserting two vertices in an edge of an SMPS T or modifying a region of T into a tetrahedron, octahedron, or icosahedron, whose vertices have a correct configuration following a 4-partite graph. These modifications may be applied to complete tripartite graphs to construct SMPS for the case of complete 4-partite graphs, which results to relationships between the cardinalities of these two graphs. Based on what was shown in this paper, it is clear that from an SMPS of a complete k-partite graph, one can construct an SMPS of a complete q-partite graph, $k < q$, by a sequence of adding vertices and edges to preserve maximal planarity. It would be interesting to find more relationships between the partite sets of these graphs, similar to the relationships found between complete tripartite graphs and complete 4-partite graphs.

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